JOURNAL OF SPACECRAFT AND ROCKETS
Vol. 37, No. 5, September–October 2000

Critical Cases of Ballistic Entry: New, Guidance-Oriented, Higher-Order Analytic Solutions

L. de-Olivé Ferreira*
Instituto de Aeronáutica e Espaço, 12228-904 São José dos Campos, Brazil
and
Nguyen X. Vinh† and Donald T. Greenwood‡
University of Michigan, Ann Arbor, Michigan 48109-2118

New, higher-order analytic solutions have recently been obtained for the critical cases of 1) ballistic skip trajectories with initial speed greater than the circular orbital speed, 2) shallow ballistic entry with entry speed less than the circular orbital speed, and 3) shallow ballistic entry from low circular orbits. Both noncircular cases admit the same set of analytic solutions, except that the supercircular case involves the regular error function of real argument, whereas in the subcircular case the solution depends on error functions of imaginary argument. Neither case now exhibits singularities in the flight-path angle, thus superseding results currently available in the literature. For the circular case a totally independent set of parametric expressions was obtained. In comparison with the numerical integration of the equations of motion, the analytic solutions exhibit a high degree of accuracy. A definite improvement has thus been achieved over equivalent solutions found in the literature. Special heed is taken of the effective-entry cases.

Nomenclature

B	=	small dimensionless parameter specifying entry altitude and physical characteristics of the vehicle
$\bar{B}$	=	B/E^*
C_D, C_L	=	coefficients of drag and lift, respectively
C_{D_0}	=	zero-lift coefficient of drag
c	=	dimensionless flight-path angle at entry
D	=	drag, N
E^*	=	maximum lift-to-drag ratio
g	=	magnitude of the acceleration as a result of gravity, m/s^2
h	=	dimensionless altitude from the reference level
K	=	induced drag factor
k	=	auxiliary dimensionless parameter
L	=	lift, N
m	=	vehicle's mass, kg
r	=	radial distance from planet's center, km
S	=	vehicle's characteristic area, m^2
t	=	time, s
u	=	dimensionless speed in terms of the kinetic energy
V	=	speed along the trajectory, km/s
x	=	initial dimensionless flight-path angle (noncircular cases)
y	=	dimensionless altitude/density variable
z	=	auxiliary dimensionless variable in the computation of the altitude
α	=	parameter defining trajectory type
β	=	inverse scale height, km^{-1}
γ	=	flight-path angle, rad
δ	=	$2(1-\alpha)$
η	=	$\bar{B}/\sqrt{(\beta r_0)}$
θ	=	range angle, rad
λ	=	normalized lift coefficient
ν	=	dimensionless speed
ρ	=	atmospheric density, kg/m^3
τ	=	transformed range angle, rad
ϕ	=	dimensionless flight-path angle
χ	=	transformed longitude, rad

Subscripts

e	=	condition at entry
0	=	reference trajectory

Received 14 April 1998; revision received 14 June 2000; accepted for publication 21 June 2000.

*Postdoctoral Fellow, Flight Dynamics Subdivision. Member AIAA.
†Professor Emeritus of Aerospace Engineering, Department of Aerospace Engineering. Member AIAA.
‡Professor Emeritus of Aerospace Engineering, Department of Aerospace Engineering. Associate Fellow AIAA.

Introduction

A NEED to improve the accuracy of Chapman's classic first-order solution[1] as applied to the guidance problem has led a number of researchers to propose new, higher-than-first-order analytic solutions for planetary entry trajectories. Most notably, Loh,[2] Yaroshevsky[3] (also see Refs. 4 and 5), and Longuski and Vinh,[6] working independently, introduced different sets of second-order solutions, none of which turned out to be completely satisfactory. Loh's integration of the equations of motion was heuristic, introducing a step lacking in mathematical justification. On the other hand, Yaroshevsky's formulation (first worked out in 1959) is plagued by a strong singularity in the flight-path angle, which is dealt with in a somewhat artificial way. Following a similar path, Longuski and Vinh integrated a system of simplified equations of motion, both numerically and analytically. Their work yields good results to within the validity of the simplifications made, but fails in the critical case of shallow entry.

Ballistic missiles typically reenter the atmosphere from subcircular orbital speeds (with the entry angle small in magnitude). Also, the abort of ascending flights of space vehicles is followed by an entry-like trajectory starting at subcircular speed. Therefore, the ability to rapidly develop accurate, reliable predictions of these types of trajectories is of paramount importance in both theory and practice. Furthermore, the current interest in aeroassist technology obviously necessitates good determination of skip trajectories. In addition, low-eccentricity, low-altitude orbits typically graze the outer layers of the atmosphere at approximately the circular orbital speed. That is the case of several classes of satellites, which, upon reaching the end of their useful lifetime, normally reenter the atmosphere somewhere along the grazing portion of their orbits.

Clearly, a need exists for good analytic solutions involving critical entry trajectories. As part of an ongoing research effort,[7,8] this paper addresses such a problem, focusing special attention on the effective-entry cases. The equations of motion are normalized via introduction of a set of Chapman-type nondimensional variables, being subsequently put in a form suitable for integration by Poincaré's analytic continuation. In previous works[2] approximate differential relations between the main variables were integrated directly. Here, on the contrary, we integrate the equations of motion in terms of the longitude. This enables the analyst to predict, for any point along the trajectory, the altitude, speed, flight-path angle, and deceleration caused by atmospheric drag as functions of the current position and the entry conditions only, instead of having to infer the position from cross plots. The resulting analytic solutions have been found to be accurate and fast for any nonthrusting, nonlifting aerospace vehicle, in any planetary atmosphere, through practically the entire range of hypersonic velocities. Indeed, for the Earth's atmosphere even at low supersonic speeds (Mach 1.6) the altitude, speed, and deceleration are predicted by the expressions herein with a level of accuracy never before achieved. These expressions are thus deemed to lend themselves well to programming on onboard computers for in-flight propagation of the nominal trajectory. The solutions are also well suited for the generation of, for example, reference trajectories for mission design and stability analysis of entry trajectories.

Dimensionless Equations of Motion

Considering Fig. 1, planar entry of a nonthrusting, lifting vehicle into a nonrotating spherical planetary atmosphere is governed by the following four equations:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = V\sin\gamma, \qquad r\frac{\mathrm{d}\theta}{\mathrm{d}t} = V\cos\gamma, \qquad \frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{D}{m} - g\sin\gamma$$

$$V\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{L}{m} - \left(g - \frac{V^2}{r}\right)\cos\gamma \tag{1}$$

Using the initial altitude as a reference, we consider a central Newtonian gravitational field and an exponential atmosphere:

$$g/g_0 = r_0^2\big/r^2, \qquad \rho/\rho_0 = \exp\{-\beta(r - r_0)\} \tag{2}$$

For the lift-drag relationship we adopt the standard parabolic polar

$$C_D = C_{D_0} + KC_L^2 \tag{3}$$

where, for hypersonic flow, C_{D_0} and K are constant. The altitude and speed variables are nondimensionalized through the following definitions:

$$h = (r - r_0)/r_0, \qquad u = V^2\big/g_0 r_0 \tag{4}$$

With the time eliminated in favor of the range angle, the equations of motion become

$$\frac{\mathrm{d}h}{\mathrm{d}\theta} = (1 + h)\tan\gamma$$

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{B(1+h)(1+\lambda^2)u\exp\{-\beta r_0 h\}}{E^*\cos\gamma} - \frac{2}{(1+h)}\tan\gamma$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\theta} = \frac{B(1+h)\lambda\exp\{-\beta r_0 h\}}{\cos\gamma} - \frac{1}{u(1+h)} + 1 \tag{5}$$

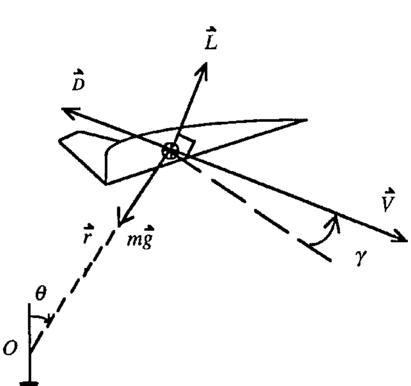

Fig. 1 Main variables of the problem. All indicated vectors are applied to the center of mass of the vehicle.

where E^* is the maximum lift-to-drag ratio and B is a small parameter specifying the entry altitude and physical characteristics of the vehicle:

$$E^* = \left(2\sqrt{KC_{D_0}}\right)^{-1}, \qquad B = \left(\tfrac{1}{2}m^{-1}\rho_0 S r_0\right)\sqrt{C_{D_0}\big/K} \tag{6}$$

Equations (5) constitute the exact equations for reentry, even if we have lift modulation. The lift control here is the normalized lift coefficient

$$\lambda = \frac{C_L}{\sqrt{C_{D_0}\big/K}} \tag{7}$$

which is defined such that, when $\lambda = 1$, the flight is at maximum lift-to-drag ratio. Analytic solutions can be obtained for constant λ, with the limiting case of $\lambda = 0$ for ballistic entry. The equations are integrated from the initial entry point at $\theta = 0$ with $h = 0$, $u = u_e$, $g = g_e$. The atmosphere is here specified by Chapman's parameter βr_0, while E^* is the performance characteristic of the vehicle. Physical characteristics of the vehicle, such as mass and size, are contained in the dimensionless parameter B, which also serves as an indication of the entry altitude.

For the integration new dimensionless variables are introduced: y for altitude, v for speed, and ϕ for flight-path angle, such that

$$y = \exp\{-\beta r_0 h\} = \rho/\rho_0, \qquad v = \ln(V_e/V)^{2/\eta}$$

$$\phi = -\sqrt{\beta r_0}\sin\gamma \tag{8}$$

where V_e is the entry speed and η is a small constant defined as

$$\eta = B\big/\left(E^*\sqrt{\beta r_0}\right) = m^{-1}\rho_0 S C_{D_0}\sqrt{r_0/\beta} \tag{9}$$

For a typical entry vehicle at 100 km in the Earth's atmosphere, η is of the order of 10^{-4}. The new variables just defined are such that, at entry, we always have $y(0) = 1$ and $v(0) = 0$. With these definitions Eqs. (5) are transformed into

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{(1+h)}{\cos\gamma}y\phi$$

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{(1+h)}{\cos\gamma}(1+\lambda^2)y - \frac{1}{(1+h)\cos\gamma}k\alpha\phi e^{\eta v}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -(1+h)B\lambda y + \frac{\cos\gamma}{(1+h)}\alpha e^{\eta v} - \cos\gamma \tag{10}$$

where $\tau = \theta\sqrt{(\beta r_0)}$ is the new independent variable and

$$k = 2E^*\big/\left(B\sqrt{\beta r_0}\right), \qquad \alpha = g_0 r_0\big/V_e^2 \tag{11}$$

Equations (10) are exact. However, our selection of dimensionless variables, combined with the fact that, for entry trajectories, $\cos\gamma \approx 1$, $1 + h \approx 1$, leads to a simpler system, viz.

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = y\phi, \qquad \frac{\mathrm{d}v}{\mathrm{d}\tau} = (1+\lambda^2)y - k\alpha\phi e^{\eta v}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -B\lambda y + \alpha e^{\eta v} - 1 \tag{12}$$

Extensive numerical integration[9,10] performed for several cases of entry trajectories has shown that the solutions to Eqs. (12) are nearly the same as those obtained from Eqs. (10).

In summary, Eqs. (12) comprise the dimensionless equations of motion for reentry, with y being the density ratio (used as the altitude variable), v the speed variable, and ϕ the flight-path angle variable, whereas τ is proportional to the distance traveled. At $\tau = 0$ we always have $y(0) = 1$ and $v(0) = 0$. The initial entry speed is now specified by the dimensionless parameter α, with $\alpha = 1$ for circular entry and $\alpha = 0.5$ for parabolic entry. The planetary atmosphere is specified by Chapman's characteristic parameter in the form of the product βr_0, with the value $\beta r_0 \approx 900$ for the Earth's atmosphere.

The vehicle physical characteristics are specified by the maximum lift-to-drag ratio E^* and the dimensionless parameter B, which also includes the effective entry altitude. The number of physical parameters required for the analysis has thus been reduced to a minimum. To generate the trajectory, it suffices to select an entry angle γ_e [with which the initial value $\phi_e = -(\beta r_0)^{1/2} \sin\gamma_e$ is evaluated] and to specify the flight program in terms of λ, the coefficient of lift.

Noncircular Cases

In this section we develop analytic solutions for both the case of ballistic skip with an initial speed greater than the local circular speed and the case of ballistic entry from a subcircular speed. Special attention is focused on the entry case. Ballistic trajectories are generally characterized by the absence of aerodynamic lift, that is, a zero normalized lift coefficient ($\lambda = 0$). Equations (12) are hence particularized to

$$\frac{dy}{d\tau} = y\phi, \qquad \frac{dv}{d\tau} = y - k\alpha\phi e^{\eta v}, \qquad \frac{d\phi}{d\tau} = \alpha e^{\eta v} - 1 \tag{13}$$

with initial conditions at $\tau = 0$

$$y(0) = 1, \qquad v(0) = 0, \qquad \phi(0) = c \tag{14}$$

where $c = -(\beta r_0)^{1/2}\sin\gamma_e > 0$ is the entry angle, whereas the entry speed is specified by selecting a value for α. The vehicle physical characteristics and the effective entry altitude are contained in the two parameters k and η. For ballistic trajectories in the Earth's atmosphere, because we know the value $\beta r_0 \approx 900$ it suffices to specify

$$\bar{B} = B/E^* = m^{-1}\rho_0 S C_{D_0} r_0 \tag{15}$$

to replace both k and η. Therefore, a ballistic trajectory is governed by the initial speed, the initial angle, and the factor $\bar{B}$ as defined in Eq. (15).

The ballistic entry mode at subcircular speed and moderate entry angle was extensively studied in the past, and fairly accurate solutions were obtained.[6] Nevertheless, a new, improved set of solutions is derived here. The ballistic mode can also be used for aeroassisted orbital transfer and planetary aerocapture. On the other hand, with supercircular initial speed and a small initial angle, the vehicle will skip out, and the accurate prediction of the exit speed and exit angle using explicit analytic expressions is of obvious interest to mission design. Furthermore, if, from a computational standpoint, in addition to being accurate such expressions are fast enough to become competitive when compared to the numerical integration of the exact equations of motion, these analytic expressions can then be used for guidance purposes advantageously.

It will become apparent in the following derivation that both the ballistic skip and the actual ballistic entry admit essentially the same set of analytic solutions, with the only notable difference residing in the evaluation of the speed. In the supercircular case the speed is given by a combination of error functions of real arguments, whereas in the subcircular case the speed depends on error functions of imaginary arguments.

Higher-order analytic solutions are obtained by observing that η is a small parameter, of the order of 10^{-4}. We seek solutions of the form

$$y = y_0 + \eta y_1 + \eta^2 y_2 + \cdots, \qquad v = v_0 + \eta v_1 + \eta^2 v_2 + \cdots$$

$$\phi = \phi_0 + \eta\phi_1 + \eta^2\phi_2 + \cdots \tag{16}$$

with initial conditions

$$y_0(0) = 1, \qquad v_0(0) = 0, \qquad \phi_0(0) = c \tag{17}$$

$$y_i(0) = 0, \qquad v_i(0) = 0, \qquad \phi_i(0) = 0, \qquad i = 1, 2, \ldots \tag{18}$$

Substituting Eqs. (16) into Eqs. (13) and equating terms of like powers of η, the following systems of various orders are obtained:

$$\frac{dy_0}{d\tau} = y_0\phi_0, \qquad \frac{dv_0}{d\tau} = y_0 - k\alpha\phi_0, \qquad \frac{d\phi_0}{d\tau} = -(1-\alpha) \tag{19}$$

$$\frac{dy_1}{d\tau} = y_0\phi_1 + y_1\phi_0, \qquad \frac{dv_1}{d\tau} = y_1 - k\alpha(\phi_0 v_0 + \phi_1)$$

$$\frac{d\phi_1}{d\tau} = \alpha v_0 \tag{20}$$

$$\frac{dy_2}{d\tau} = y_0\phi_2 + y_1\phi_1 + y_2\phi_0$$

$$\frac{dv_2}{d\tau} = y_2 - k\alpha\left(\phi_0 v_1 + \frac{1}{2}\phi_0 v_0^2 + \phi_1 v_0 + \phi_2\right)$$

$$\frac{d\phi_2}{d\tau} = \alpha\left(v_1 + \frac{1}{2}v_0^2\right) \tag{21}$$

Because the solution for ϕ_0 is linear in τ, it is convenient to use this function as the new independent variable. For ease of notation, let

$$x = \phi_0, \qquad \delta = 2(1-\alpha) \tag{22}$$

Notice that, in the supercircular case, $\alpha < 1$, implying $\delta > 0$. In the subcircular case, however, $\alpha > 1$, and $\delta < 0$.

Introducing the preceding definitions into system (19), the equations for the first-order solution become

$$\frac{dy_0}{dx} = -\left(\frac{2}{\delta}\right)y_0 x, \qquad \frac{dv_0}{dx} = -\left(\frac{2}{\delta}\right)(y_0 - k\alpha x) \tag{23}$$

With the initial value $x(0) = c$, the solution is readily obtained as

$$y_0 = \exp\{(c^2 - x^2)/\delta\} \tag{24}$$

$$v_0 = -(k\alpha/\delta)(c^2 - x^2) + \sqrt{\pi/\delta}\exp\{c^2/\delta\} \times\left[\operatorname{erf}\left(c/\sqrt{\delta}\right) - \operatorname{erf}\left(x/\sqrt{\delta}\right)\right] \tag{25}$$

where the error function is defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\zeta^2}\, d\zeta \tag{26}$$

and z is a complex variable. Evidently, with $\delta > 0$ all of the square roots contained in Eq. (25) are real, and the error functions therein are of the ordinary type, in the supercircular case. On the other hand, in the subcircular case, with $\delta < 0$, the square roots in Eq. (25) are imaginary so that the arguments of the error functions are now imaginary. To perform the evaluation of Eq. (25) when $\delta < 0$, we resort to a series approximation for the complex error function as listed in Ref. 11 and first published by Salzer.[12] Upon applying that series approximation to the problem of ballistic entry (and using the dimensionless variables just defined), the analytic expression for the speed during ballistic entry becomes

$$v_0 = -\frac{k\alpha}{\delta}(c^2 - x^2) + \frac{1}{\delta\sqrt{\pi}}\exp\left\{\frac{c^2}{\delta}\right\}\left\{(c-x) + 2\sqrt{-\delta}\sum_{n=1}^{\infty}\left(\frac{1}{ne^{n^2/4}}\right)\left[\sinh\left(\frac{nc}{\sqrt{-\delta}}\right) - \sinh\left(\frac{nx}{\sqrt{-\delta}}\right)\right]\right\} \tag{27}$$

an expression first obtained in Ref. 9.

Although (conditional) convergence of this series can be shown by means of a judicious concatenation of the comparison test and D'Alembert's ratio test, analytic determination of the radius of convergence has proven a rather difficult task. Nevertheless, it has been determined that, to within the double-precision arithmetic available on present-day computers, as long as the small-angle hypothesis is preserved good results will be obtained by utilizing a refined increment of the independent variable x and 25 terms in the series, what seems to be in agreement with Salzer's recommendation. Utilizing error analysis, that author concluded that 24 terms should lead to maximum accuracy.

From the last equation in system (19), ϕ_0 is related to the distance traveled by

$$\phi_0 = c - (\delta/2)\tau \tag{28}$$

Although ϕ_0 is the first-order solution for the flight-path angle, we use it in the higher-order solution simply as the monotonically varying independent variable by putting $x = \phi_0$. We thus rewrite Eqs. (20) and (21) as

$$\frac{d\phi_1}{dx} = -\left(\frac{2}{\delta}\right)\alpha v_0, \qquad \frac{dy_1}{dx} = -\left(\frac{2}{\delta}\right)(y_0\phi_1 + y_1 x)$$

$$\frac{dv_1}{dx} = -\left(\frac{2}{\delta}\right)[y_1 - k\alpha(\phi_1 + v_0 x)] \tag{29}$$

$$\frac{d\phi_2}{dx} = -\left(\frac{2}{\delta}\right)\alpha\left(v_1 + \frac{1}{2}v_0^2\right)$$

$$\frac{dy_2}{dx} = -\left(\frac{2}{\delta}\right)(y_0\phi_2 + y_1\phi_1 + y_2 x)$$

$$\frac{dv_2}{dx} = -\left(\frac{2}{\delta}\right)\left[y_2 - k\alpha\left(\phi_2 + \phi_1 v_0 + v_1 x + \frac{1}{2}v_0^2 x\right)\right] \tag{30}$$

The second-order solutions are obtained by integrating Eqs. (29) with null initial values for all of the variables. Then, substituting the first- and second-order solutions into Eqs. (30), the same process can be used to obtain third-order solutions, and so on.

Initially, using the solution for v_0 in the first of Eqs. (29), $\phi_1(x)$ is obtained by quadrature. Direct integration by parts with an efficient use of the existing differential relations enables one to cast the integration of a long expression into some simple integrals. Hence, with the initial condition $\phi_1(0) = 0$

$$\phi_1 = (2\alpha/\delta)(y_0 - v_0 x - 1) - (4k\alpha^2/3\delta^2)(c^3 - x^3) \tag{31}$$

Next, observing that the homogeneous equation for y_1 (and later on, for $y_2, y_3, \ldots$) is of the same form as the equation for y_0, we put

$$y_i = y_0 z_i, \qquad i = 1, 2, \ldots \tag{32}$$

Then, the equation for y_1 becomes

$$\frac{dz_1}{dx} = -\left(\frac{2}{\delta}\right)\phi_1 \tag{33}$$

with initial condition $z_1(0) = 0$. The remark just made about integration by parts with successive transformations of the integrals involved applies here as well. After using this procedure to obtain z_1, the final result for y_1 is

$$\begin{aligned} y_1 = {} & -(2\alpha/\delta^2)y_0 x\left[y_0 - v_0 x - 1 - (4k\alpha/3\delta)(c^3 - x^3)\right] \\ & - (k\alpha^2/\delta^2)y_0[\delta^{-1}(c^4 - x^4) - (c^2 - x^2)] \\ & + (\alpha/\delta)y_0[v_0 - (2/\delta)(c - x)] \end{aligned} \tag{34}$$

Substituting v_0, ϕ_1, and y_1 into the last equation in system (29), we obtain v_1 by quadrature, using the same technique of integration by parts:

$$\begin{aligned} v_1 = {} & -(2\alpha/\delta^2)y_0\left[y_0 - v_0 x - 1 - (4k\alpha/3\delta)(c^3 - x^3)\right] \\ & + (4k\alpha^2/\delta^2)x\left[y_0 - v_0 x - 1 - (2k\alpha/3\delta)(c^3 - x^3)\right] \\ & + (2\alpha/\delta^2)y_0[(k\alpha/2\delta)x^3 - (k/4)(4 - \alpha)x + 1] \\ & - (2\alpha/\delta^2)[(k\alpha/2\delta)c^3 - (k/4)(4 - \alpha)c + 1] \\ & - (2\alpha/\delta)[(k\alpha/2\delta^2)c^4 - (k\alpha/2\delta)c^2 + \delta^{-1}c + (k/8)(4 - \alpha)] \\ & \times\left[v_0 + (k\alpha/\delta)(c^2 - x^2)\right] + (\alpha/\delta)v_0^2 + (k\alpha/\delta)v_0 x^2 \\ & + (k^2\alpha^2/2\delta^2)(c^4 - x^4) \end{aligned} \tag{35}$$

These second-order solutions are valid for both the cases of super- and subcircular entry. Particularization to either case requires only that the appropriate solution for v_0 [Eq. (25) or Eq. (27)] be used in the second-order expressions. Except for the computation of v_0, the difference between super- and subcircular entry has no major bearing on the solution process, with the same set of analytic expressions resulting for altitude, flight-path angle, and speed.

In addition, when planning any entry-like trajectory it is of interest to assess the deceleration caused by aerodynamic drag. This deceleration is given by

$$\frac{a}{g} = -g^{-1}\left(\frac{dV}{dt}\right) = \frac{1}{2}m^{-1}g^{-1}\rho SC_D V^2 \tag{36}$$

With $g \approx g_0$ and using the dimensionless variables already defined, the preceding expression becomes

$$a/g_0 = \tfrac{1}{2}\bar{B}\alpha^{-1}y\exp\{-\eta v\} = \tfrac{1}{2}\bar{B}yu \tag{37}$$

We stress the fact that it suffices to provide a value for $\bar{B}$ as defined in Eq. (15).

As reported in Ref. 8, for a given value of the entry angle (e.g., $\gamma_e = -3$ deg), the higher the entry speed, the closer the analytic solutions will match the numerical solution. As lower values of the entry speed are chosen, the agreement between analytic and numerical becomes less and less satisfactory until, for entry speeds below 1.1 times the circular orbital speed, the present set of solutions no longer yields reliable predictions. Furthermore, it has been found[9] that, in order to obtain acceptable results corresponding to such low entry speeds, the entry angle must be tightly controlled, remaining within the $-2\ \text{deg} \le \gamma_e \le 0$ deg range. For that reason an ad hoc analytic solution becomes necessary for entry at the circular speed.[9] On the other hand, for a given value of the entry speed (say, the parabolic entry speed, $u_e = 2$), by increasing the absolute value of the entry angle, some accuracy is lost, as the small angle hypothesis is violated. Again, the best results will be obtained by remaining within the range of entry angles just mentioned.

The solutions derived up to this point contain no singularity in the flight-path angle, thereby allowing the propagation of shallow entry trajectories. They do, however, embody a discontinuity in the entry speed, corresponding to the case of entry from circular orbital speed, for which, as we have already pointed out, an altogether different set of analytic solutions will be required.

The skip case having been extensively explored in Refs. 7 and 8, we move on to focus on the accuracy of the new solutions when applied to the effective-entry cases, beginning with the subcircular entry. To that end, refer to Figs. 2–4. (In all figures the solid lines correspond to the numerical solution, whereas the dashed and dotted lines represent the analytic solutions of various orders.)

Figures 2–4 reflect a typical Earth reentry maneuver, injected at $u_e = 0.5 (\approx$ Mach 16.3), corresponding to $V/V_c \approx 0.7$. As indicated by the plot of altitude vs dimensionless speed, Fig. 2, the vehicle initially plummets steeply into the atmosphere at high speed, covering about half the total altitude range ($|\Delta h| = 0.008$, that is, altitude variation $\approx$ 51.8 km). By then, the vehicle is deep enough in the atmosphere for the density to become a relevant factor, inducing a sensible increase in aerodynamic drag. The deceleration is now large (reaching a peak at approximately $V/V_c \approx 0.4$, $h = -0.011$, altitude $\approx$ 28.7 km), so that for the remainder of the trajectory the altitude varies much more slowly until the entry phase is completed at about $h = -0.012$, that is, altitude $\approx$ 22.3 km, and a speed of $u_f = 0.005$ ($V/V_c \approx 0.07 \approx$ Mach 1.6). The flight-path angle initially decreases steeply at high altitude. Then, through the largest portion of the deceleration phase, γ remains nearly constant. Finally, toward the end of the maneuver it begins to decrease fast again. The behavior of the trajectory can be explained by the fact that the entry point at subcircular speed and small entry angle is nearly the apogee of a highly eccentric elliptic orbit and, as such, at the beginning, the radial distance undergoes a fast decrease.

As in the supercircular case, here, too, one can see that, along the entire trajectory, the second-order solution gives a much better approximation than the first-order, actually overlapping with the

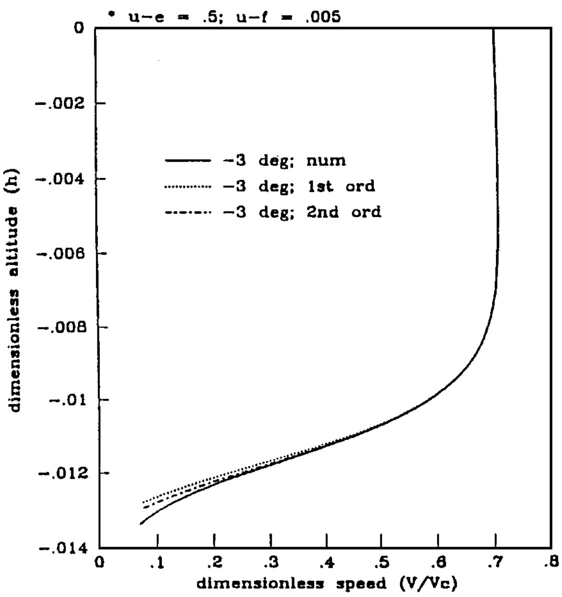

Fig. 2 Variation of the altitude during ballistic entry at subcircular speed (h vs V/V_c; $u_e = 0.5$).

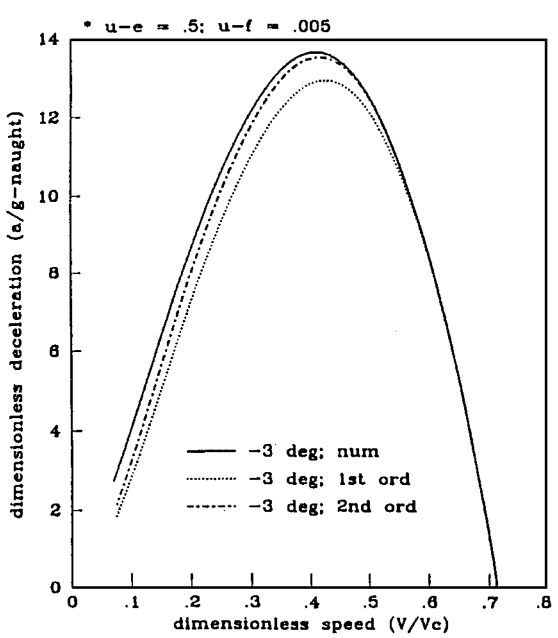

Fig. 3 Deceleration caused by drag during ballistic entry at subcircular speed (a/g vs V/V_c; $u_e = 0.5$).

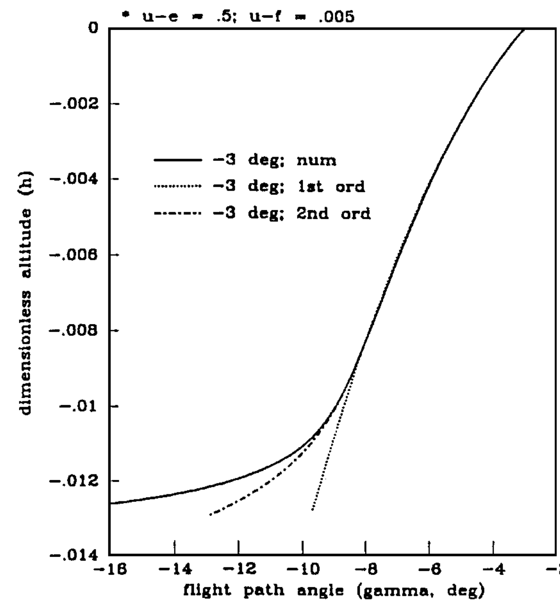

Fig. 4 Variation of the altitude for ballistic entry at subcircular speed (h vs γ; $u_e = 0.5$).

numerical solution through the first half of the trajectory. The accuracy is good. For instance, at the end of the entry phase the second-order prediction of altitude misses the exact value by a mere 3.7%. In fact, the analytic predictions of altitude, speed, and deceleration are very accurate. The predictions of the flight-path angle are a little less satisfactory, but still within acceptable boundaries. Unlike the supercircular case, now the larger $|\gamma_e|$, the greater the accuracy.

In general, the equations of motion for each new, higher order depend on the solutions for ϕ, y, and v of all lower orders. Such being the case, by efficiently exploiting differential relationships extracted from the lower-order equations of motion those for each new order can be ultimately recast in terms of the first-order solution, thus enabling the interested researcher to seek higher-order solutions as needed.

Circular Case

When the entry is injected at a nominal circular speed, $u_e = 1$, $\alpha = 1$. Equations (19–21) then become, respectively,

$$\frac{\mathrm{d}y_0}{\mathrm{d}\tau} = y_0\phi_0 \tag{38a}$$

$$\frac{\mathrm{d}v_0}{\mathrm{d}\tau} = y_0 - k\phi_0 \tag{38b}$$

$$\frac{\mathrm{d}\phi_0}{\mathrm{d}\tau} = 0 \tag{38c}$$

$$\frac{\mathrm{d}y_1}{\mathrm{d}\tau} = y_0\phi_1 + y_1\phi_0 \tag{39a}$$

$$\frac{\mathrm{d}v_1}{\mathrm{d}\tau} = y_1 - k(\phi_0 v_0 + \phi_1) \tag{39b}$$

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}\tau} = v_0 \tag{39c}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}\tau} = y_0\phi_2 + y_1\phi_1 + y_2\phi_0 \tag{40a}$$

$$\frac{\mathrm{d}v_2}{\mathrm{d}\tau} = y_2 - k\left(\phi_0 v_1 + \frac{1}{2}\phi_0 v_0^2 + \phi_1 v_0 + \phi_2\right) \tag{40b}$$

$$\frac{\mathrm{d}\phi_2}{\mathrm{d}\tau} = v_1 + \frac{1}{2}v_0^2 \tag{40c}$$

From Eq. (38c) it is evident that, in the circular case, ϕ_0 is a constant

$$\phi_0 = c \tag{41}$$

Thus, to obtain the complete first-order solution it is necessary only to integrate the equations for y_0 and v_0. It is convenient to put

$$\chi = c\tau = c\theta\sqrt{\beta r_0} \tag{42}$$

where χ is essentially the range angle, a monotonically increasing variable with zero initial value. It bears no connection to the variable x used in the solutions for the noncircular cases. Then

$$\frac{\mathrm{d}y_0}{\mathrm{d}\chi} = y_0, \qquad \frac{\mathrm{d}v_0}{\mathrm{d}\chi} = c^{-1}y_0 - k \tag{43}$$

These can be immediately integrated to give

$$y_0 = e^{\chi} \tag{44a}$$

$$v_0 = c^{-1}(y_0 - 1) - k\chi \tag{44b}$$

With the variable χ the equations for y_1 and y_2 become, respectively,

$$\frac{dy_1}{d\chi} = y_1 + c^{-1} y_0 \phi_1 \tag{45}$$

$$\frac{dy_2}{d\chi} = y_2 + c^{-1} y_0 \phi_2 + c^{-1} y_1 \phi_1 \tag{46}$$

Before attempting to integrate either one, however, it is convenient to make the transformation $y_i = y_0 z_i$, $i = 1, 2, \ldots$, once more, to obtain

$$\frac{dz_1}{d\chi} = c^{-1} \phi_1 \tag{47}$$

$$\frac{dz_2}{d\chi} = c^{-1} \phi_2 + c^{-1} z_1 \phi_1 \tag{48}$$

thus reducing the integration of the altitude variables to a series of quadratures. In addition, because $y_0 = \exp\{\chi\}$, $dy_0 = y_0\, d\chi$, and we are frequently considering the integral

$$I = \int y_0 P(\chi)\, d\chi \tag{49}$$

where $P(\chi)$ is a polynomial in χ. Integrating by parts,

$$I = y_0[P(\chi) - P'(\chi) + P''(\chi) - P'''(\chi) + \cdots] \tag{50}$$

Using χ as the independent variable, we are now ready to start the derivation of the second-order solution.

From Eqs. (39c) and (44b)

$$\frac{d\phi_1}{d\chi} = c^{-1} v_0 = c^{-2}[(y_0 - 1) - kc\chi] \tag{51}$$

so that

$$\phi_1 = c^{-2}\left[(y_0 - 1) - \chi - (k/2)c\chi^2\right] \tag{52}$$

With this into Eq. (47),

$$\frac{dz_1}{d\chi} = c^{-1}\phi_1 = c^{-3}\left[(y_0 - 1) - \chi - \left(\frac{k}{2}\right)c\chi^2\right] \tag{53}$$

which, with the initial condition $z_1(0) = 0$, yields

$$z_1 = c^{-3}\left[(y_0 - 1) - \chi - \tfrac{1}{2}\chi^2 - (k/6)c\chi^3\right] \tag{54}$$

and therefore, by Eq. (32),

$$y_1 = c^{-3} y_0\left[(y_0 - 1) - \chi - \tfrac{1}{2}\chi^2 - (kc/6)\chi^3\right] \tag{55}$$

Finally, for v_1 we consider Eq. (39b) transformed by Eqs. (41) and (42),

$$\frac{dv_1}{d\chi} = c^{-1} y_1 - k v_0 - kc^{-1}\phi_1 \tag{56}$$

leading to

$$\begin{aligned} v_1 = {} & \tfrac{1}{2}c^{-4}(y_0 - 1)\left(y_0 - 1 - 2kc^3\right) \\ & - c^{-4} y_0 \chi\left[kc + \tfrac{1}{2}(1 - kc)\chi + (kc/6)\chi^2\right] \\ & + kc^{-3}\chi\left[1 + c^2 + \tfrac{1}{2}(1 + kc^3)\chi + (kc/6)\chi^2\right] \end{aligned} \tag{57}$$

Equations (52), (55), and (57) constitute the complete second-order analytic solution for entry trajectories in the circular case.

Proceeding in a similar manner, upon resorting to the ancillary transformations (41) and (42), Eq. (40) can also be integrated analytically to produce a third-order solution. It can then be shown that the third-order analytic expression for the flight-path angle is

$$\begin{aligned} \phi_2 = {} & \tfrac{1}{4}c^{-5}(1 + c^2)\left(y_0^2 - 1\right) - c^{-5}(2 - 3kc + c^2)(y_0 - 1) \\ & + c^{-5} y_0 \chi\left[(1 - 3kc - kc^3) - \tfrac{1}{2}(1 - 2kc)\chi - (kc/6)\chi^2\right] \\ & + \tfrac{1}{2}c^{-5}\chi[(1 + c^2 + 2kc^3) + kc(1 + 2c^2)\chi \\ & + (kc/3)(1 + 2kc^3)\chi^2 + (k^2c^2/12)\chi^3] \end{aligned} \tag{58}$$

Next, as Eqs. (47) and (48) imply

$$z_2 = \frac{1}{2} z_1^2 + c^{-1}\int \phi_2\, d\chi \tag{59}$$

once again making use of Eq. (32) we find that the third-order expression for the nondimensional altitude/density variable is

$$\begin{aligned} y_2 = {} & \tfrac{1}{8}c^{-6}(5 + c^2) y_0^2 (y_0 - 1) \\ & - \tfrac{1}{8}c^{-6}(35 - 72kc + 7c^2 - 8kc^3) y_0 (y_0 - 1) \\ & + c^{-6} y_0^2 \chi\left\{(1 - 6kc - kc^3) - [1 - (3kc/2)]\chi - (kc/3)\chi^2\right. \\ & + \tfrac{1}{2}c^{-6} y_0 \chi\left\{\left[\tfrac{11}{2} - 6kc + (3c^2/2)\right] + \left(\tfrac{5}{2} + \tfrac{1}{2}c^2 + kc^3\right)\chi\right. \\ & + [1 + (2kc/3) + (2kc^3/3)]\chi^2 \\ & + \left[\tfrac{1}{4} + (5kc/12) + (k^2c^4/6)\right]\chi^3 \\ & \left. + (kc/6)[1 + (kc/10)]\chi^4 + (k^2c^2/36)\chi^5\right\} \end{aligned} \tag{60}$$

As for the speed, from Eq. (40b) we obtain

$$\frac{dv_2}{d\chi} = c^{-1} y_2 - k\left(v_1 + \frac{1}{2}v_0^2\right) - kc^{-1}(v_0\phi_1 + \phi_2) \tag{61}$$

so that, in view of Eqs. (40c), (41), (42), and (59), integration yields the third-order formula for the nondimensional speed variable, which, in final form, reads

$$v_2 = (k/2)z_1^2 - kz_2 - kc\phi_2 - (k/2)\phi_1^2 + c^{-7} I_{v_2}(\chi) \tag{62}$$

where

$$\begin{aligned} I_{v_2}(\chi) = {} & c^6 \int y_2\, d\chi = \frac{1}{24}(5 + c^2)\left(y_0^3 - 1\right) \\ & - \frac{1}{2}\left[6 - 13kc + c^2 - \left(\frac{3kc^3}{2}\right)\right]\left(y_0^2 - 1\right) \\ & + \frac{1}{8}(33 - 104kc + 5c^2 - 16kc^3 + 72k^2c^2 + 16k^2c^4)(y_0 - 1) \\ & + \frac{1}{2} y_0^2 \chi\left[(2 - 8kc - kc^3) - (1 - 2kc)\chi - \left(\frac{kc}{3}\right)\chi^2\right] \\ & + \frac{1}{2} y_0 \chi\left\{2\left(\frac{1}{4} + \frac{1}{4}c^2 + 4kc + kc^3 - 9k^2c^2 - 2k^2c^4\right)\right. \\ & + \left(\frac{5}{2} + \frac{1}{2}c^2 - 7kc - kc^3 + 9k^2c^2 + 2k^2c^4\right)\chi \\ & + \left(\frac{kc}{3}\right)(7 + 2c^2 - 9kc - 2kc^3)\chi^2 \\ & + \left[\frac{1}{4} - \left(\frac{5kc}{12}\right) + \left(\frac{3k^2c^2}{4}\right) + \left(\frac{k^2c^4}{6}\right)\right]\chi^3 \\ & \left. + \left(\frac{kc}{6}\right)\left[1 - \left(\frac{9kc}{10}\right)\right]\chi^4 + \left(\frac{k^2c^2}{36}\right)\chi^5\right\} \end{aligned} \tag{63}$$

another result first presented in Ref. 9. Equations (58), (60), (62), and (63) represent the complete third-order analytic solution for entry trajectories in the circular case. The solutions of various orders are compared in Figs. 5–8.

Entry at circular speed displays a somewhat different behavior than the subcircular case. Whereas the altitude follows the same general trend as before, the flight-path angle does not. No steep initial variation of γ is observed. Instead, it varies quite slowly through the first two-thirds of the trajectory. Only toward the end of the entry does the rate of variation of γ increase significantly. A similar behavior is noticed in the speed. On the other hand, the deceleration displays very much the same trend as in the subcircular case. Although the analytic solutions for the circular case are quite different than the noncircular cases, comparison of the various solutions again shows very clearly that the higher the order, the greater the accuracy, with the third-order yielding very good results indeed. Also, like the subcircular case it is noted that, as $|\gamma_e|$ increases, the agreement between analytic and numerical solutions improves. Once again, while the results for altitude, speed, and deceleration are excellent, those for γ are only moderate. A fourth-order solution might, in theory, have the potential to solve the problem.

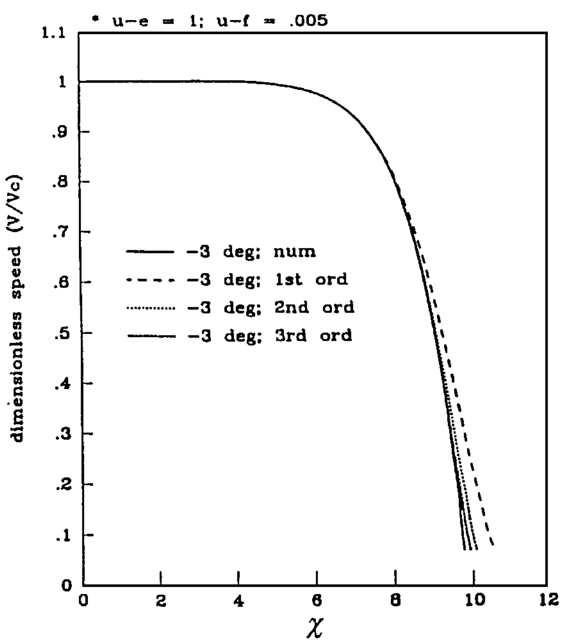

Fig. 5 Speed-longitude profile in ballistic entry at circular speed.

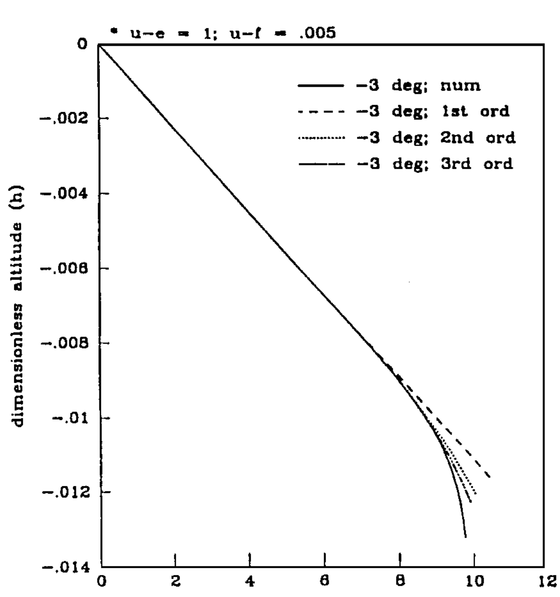

Fig. 6 Altitude-longitude profile in ballistic entry at circular speed.

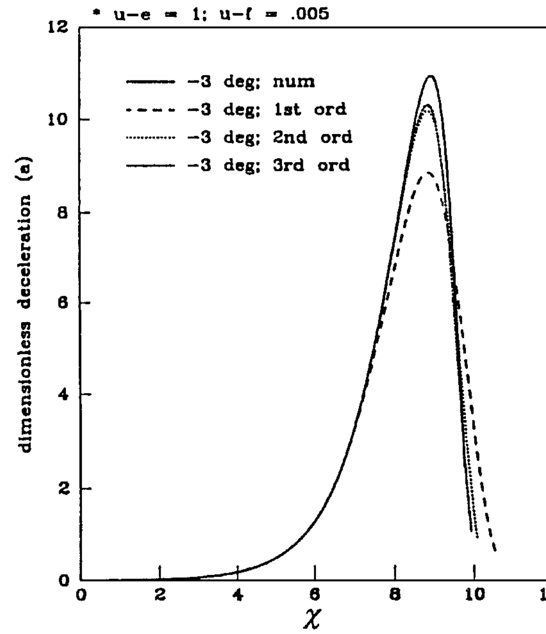

Fig. 7 Deceleration-longitude profile in ballistic entry at circular speed.

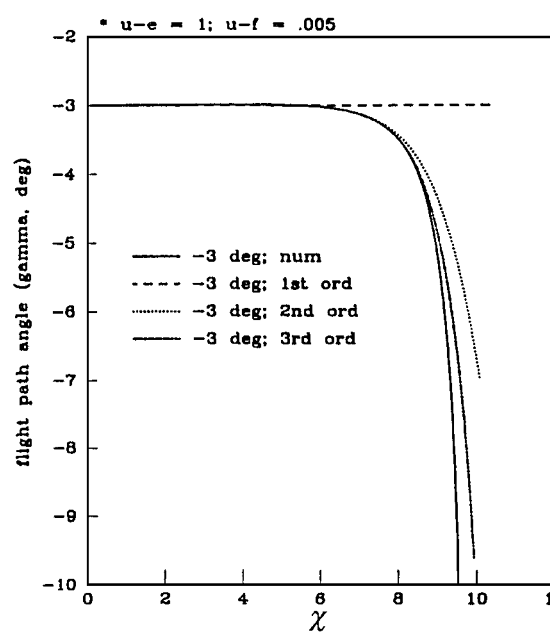

Fig. 8 Flight-path angle vs dimensionless longitude in ball entry at circular speed.

Conclusions

Regarding three classes of critical reentry like maneuvers, two new sets of guidance-oriented, higher-order analytic solutions have been introduced. One set applies to both ballistic skip at supercircular speed and shallow ballistic entry at subcircular speed, and the other applies to shallow ballistic entry from low circular orbits.

For planar entry into a nonrotating, spherical planetary atmosphere, we have normalized the equations of motion to a form suitable for integration via Poincaré's analytic continuation. The solutions obtained have been found to be accurate and fast, being also adequate for use in mission design and stability analysis. Explicit relationships between altitude, speed, flight-path angle, and distance traveled are given.

Acknowledgment

The first author is grateful to FAPESP, the Research-Supporting Foundation of the State of São Paulo, Brazil, for support for this research through Postdoctoral Fellowship 98/12640-2.

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G. D. Gamble
Associate Editor